Probability Theory
Random Variables and Distributions

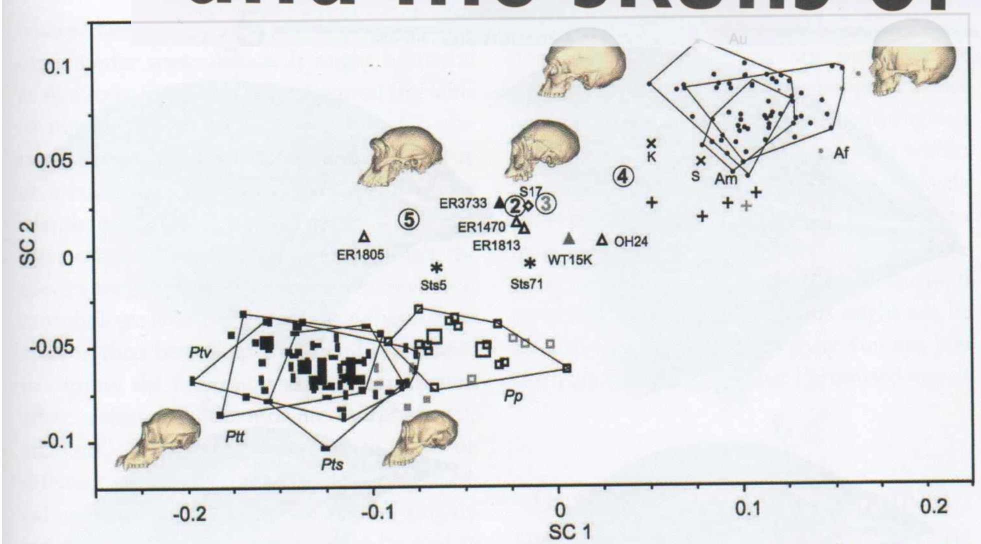
Rob Nicholls

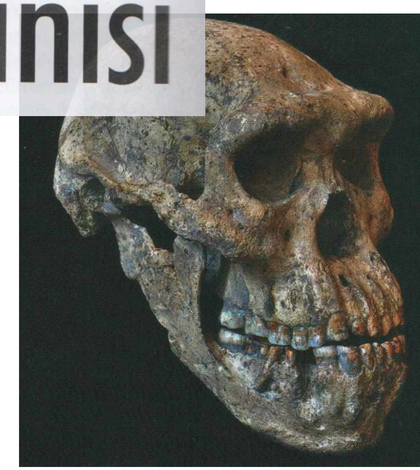
MRC LMB Statistics Course 2014

Introduction

Introduction

Early human evolution and the skulls of Dmanisi





Significance magazine (December 2013) Royal Statistical Society

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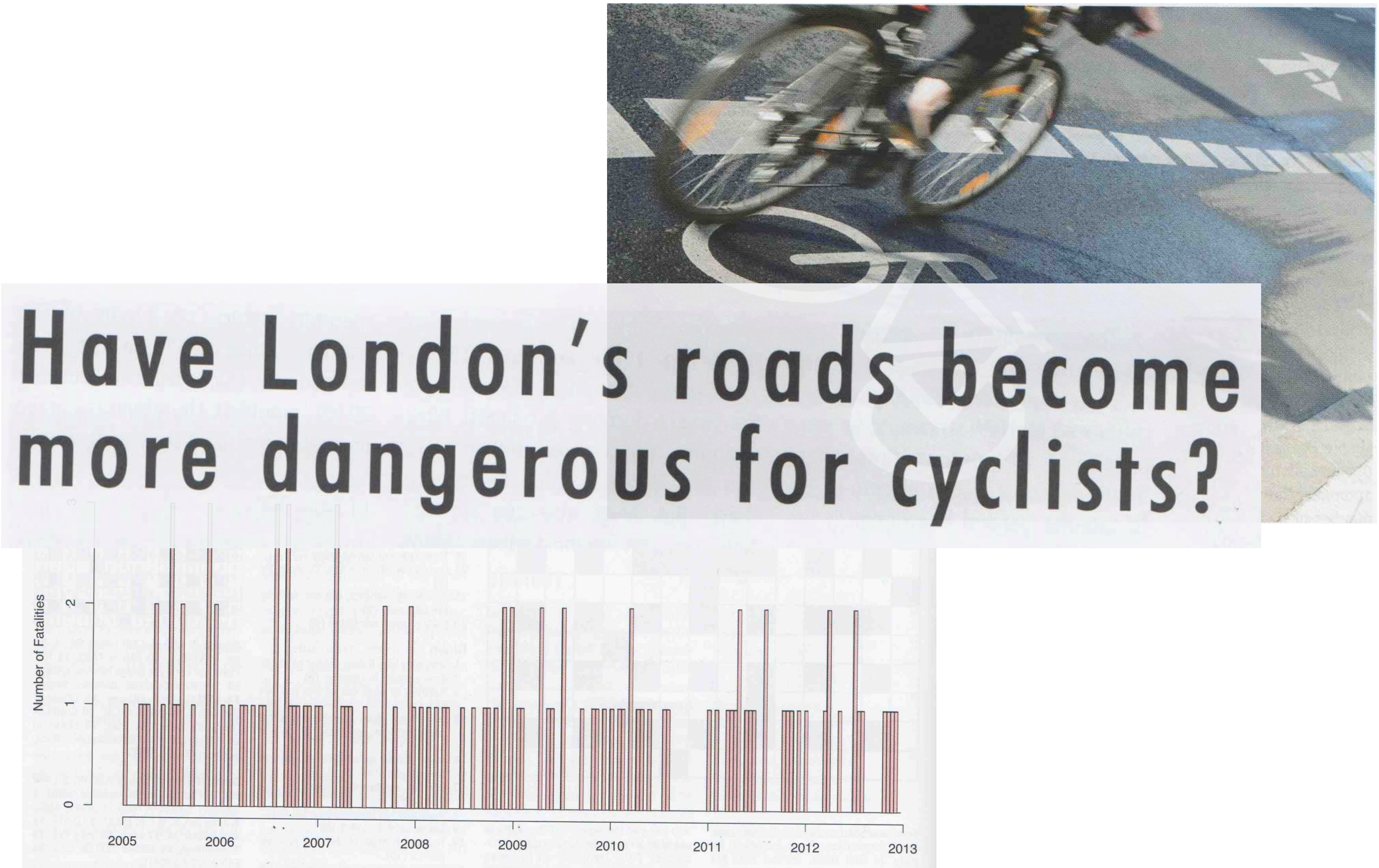

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Contents

Set Notation

A set is a collection of objects, written using curly brackets {}

If *A* is the set of all outcomes, then:

$$A = \{heads, tails\}$$

$$A = \{one, two, three, four, five, six\}$$

A set does not have to comprise the full number of outcomes

E.g. if A is the set of dice outcomes no higher than three, then:

$$A = \{one, two, three\}$$

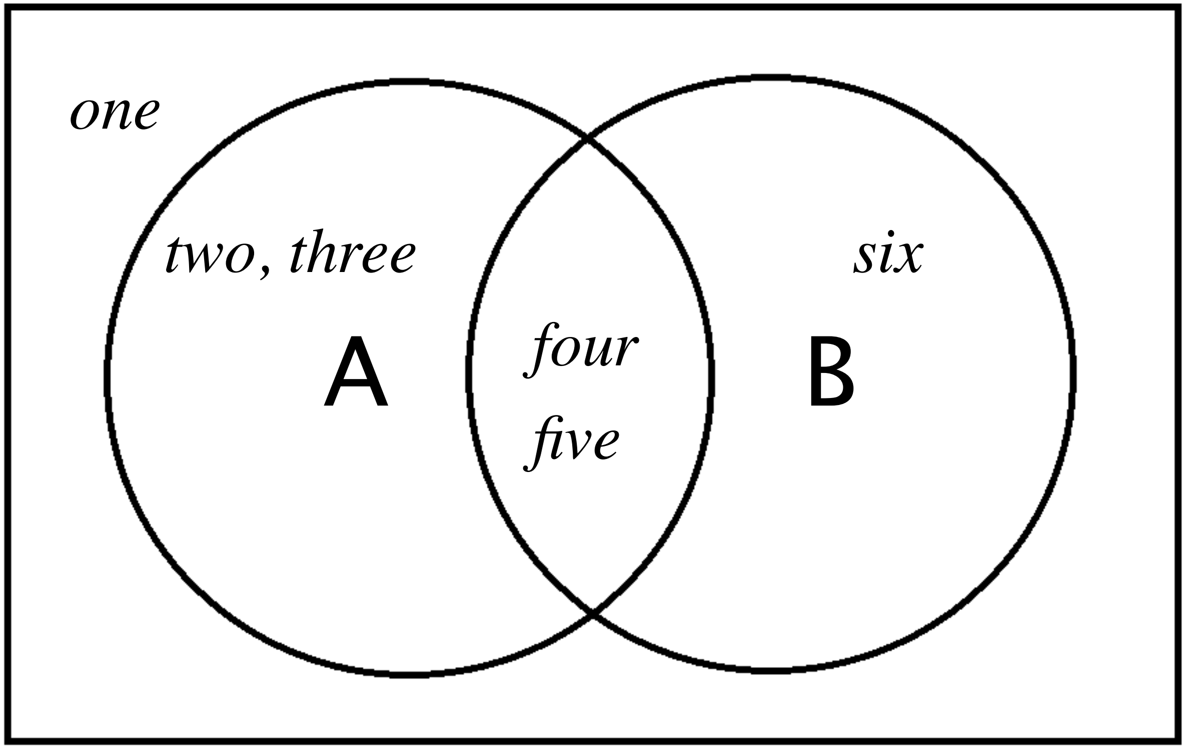
Set Notation

If A and B are sets, then:

A'	Complement – everything but A
$A \cup B$	Union (or)
$A \cap B$	Intersection (and)
$A \setminus B$	Not
$\varnothing$	Empty Set

Set Notation

Venn Diagram:



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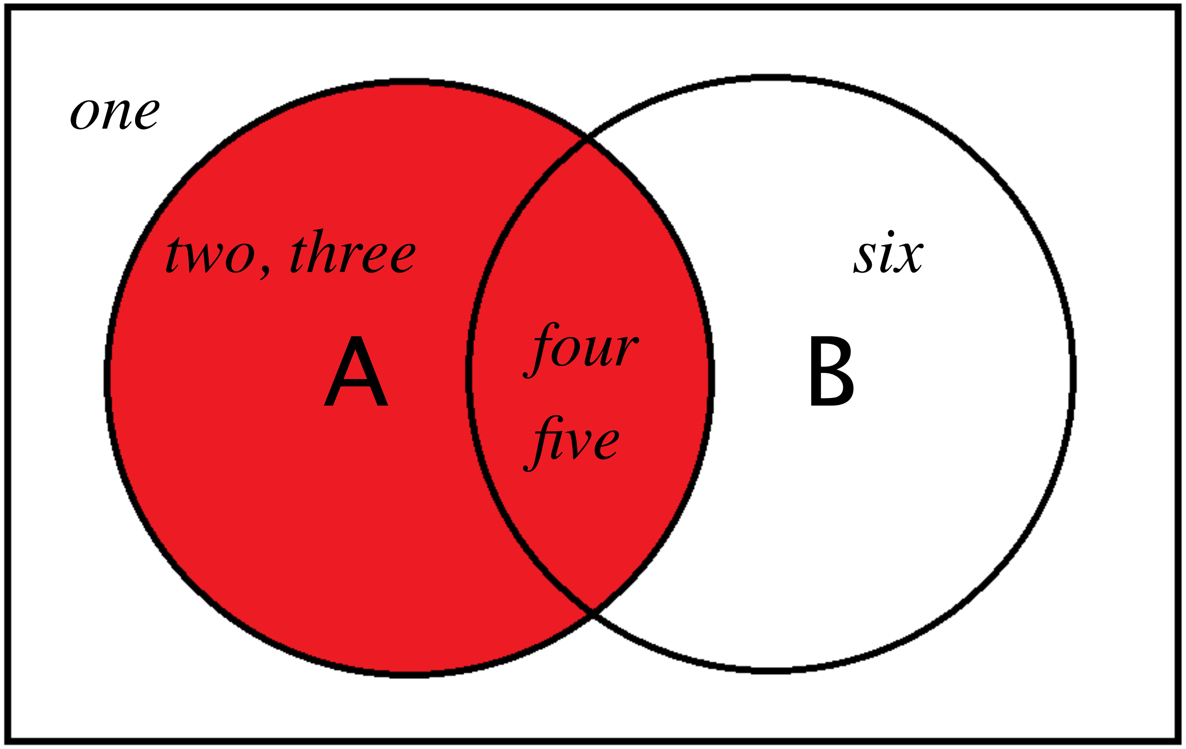
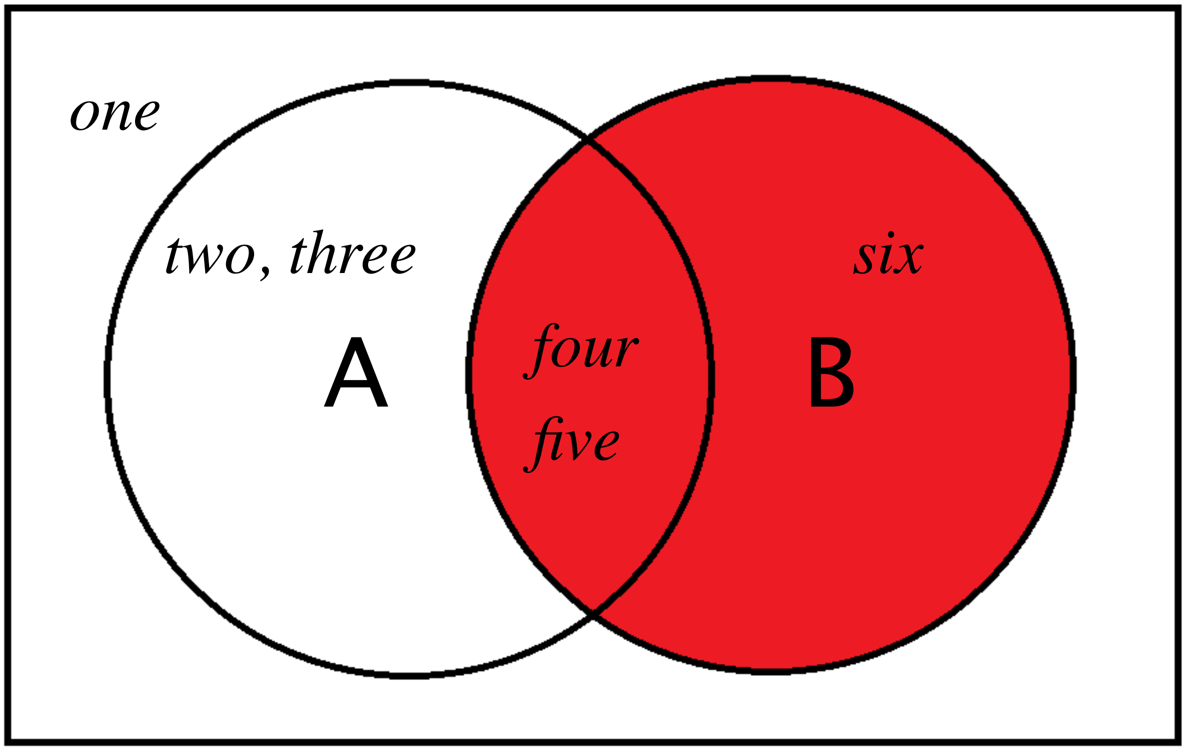


$$A = \{two, three, four, five\}$$

Set Notation

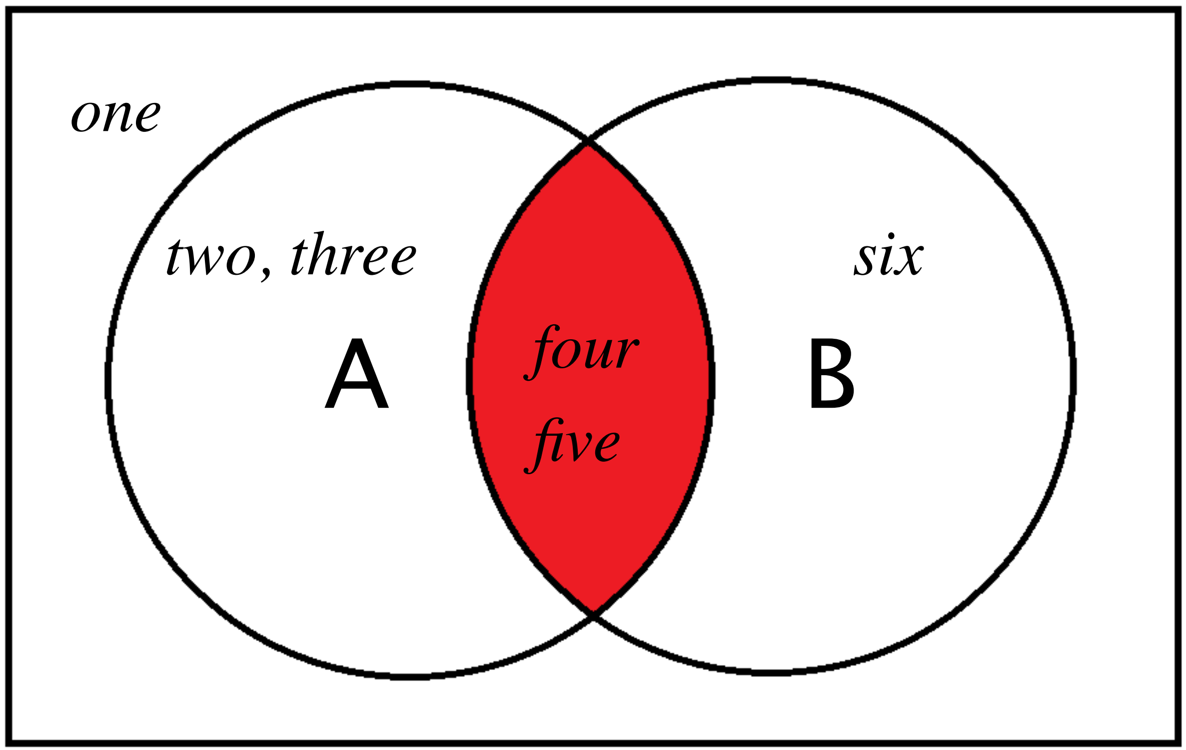
Venn Diagram:



$$B = \{four, five, six\}$$

Set Notation

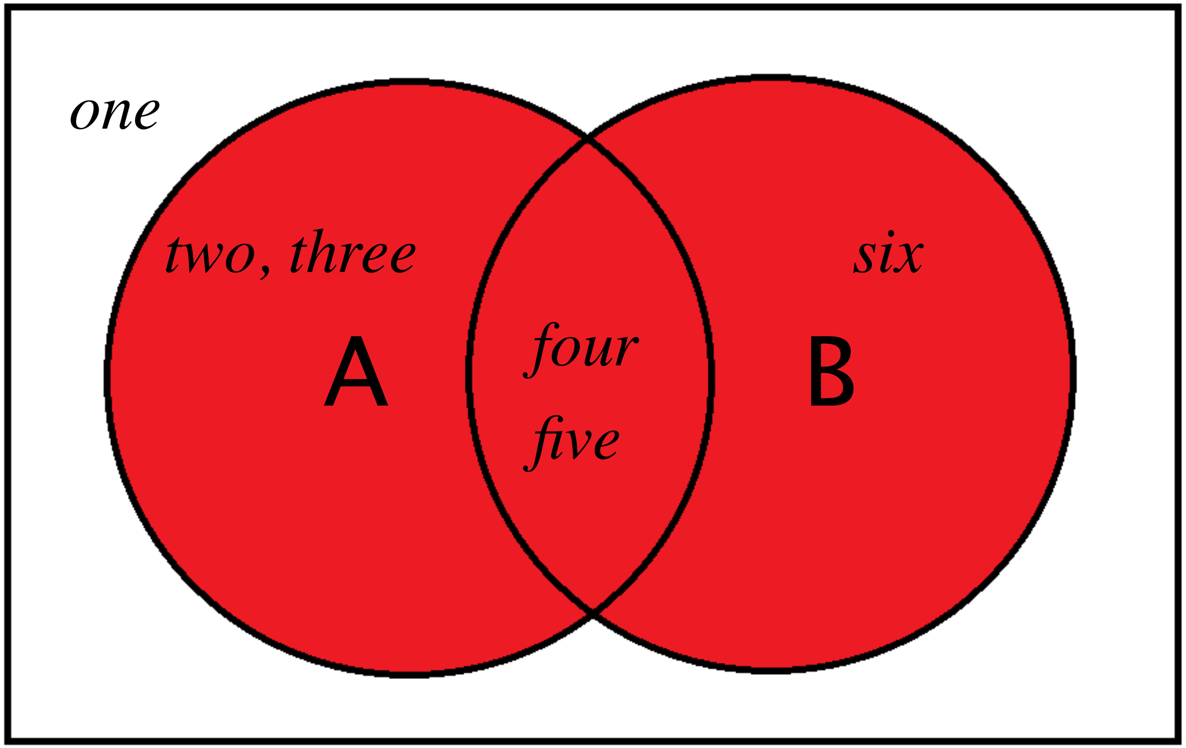
Venn Diagram:



$$A \cap B = \{four, five\}$$

Set Notation

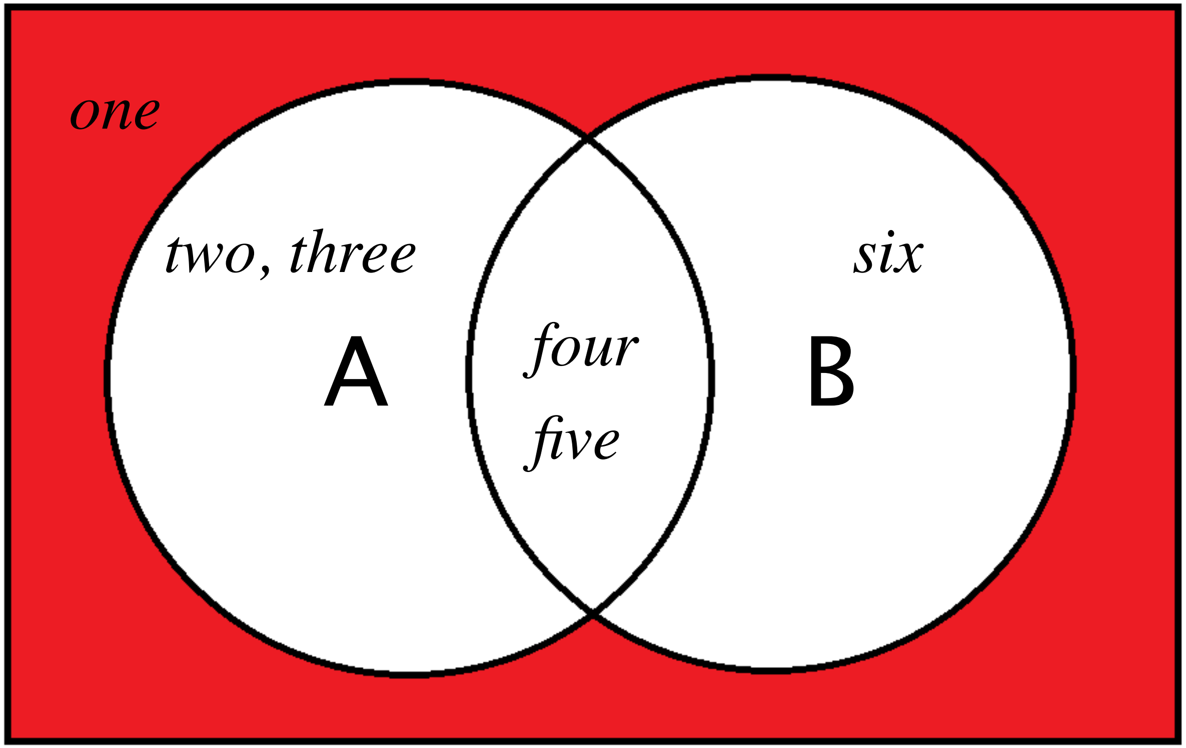
Venn Diagram:



$$A \cup B = \{two, three, four, five, six\}$$

Set Notation

Venn Diagram:



$$(A \cup B)' = \{one\}$$

Set Notation

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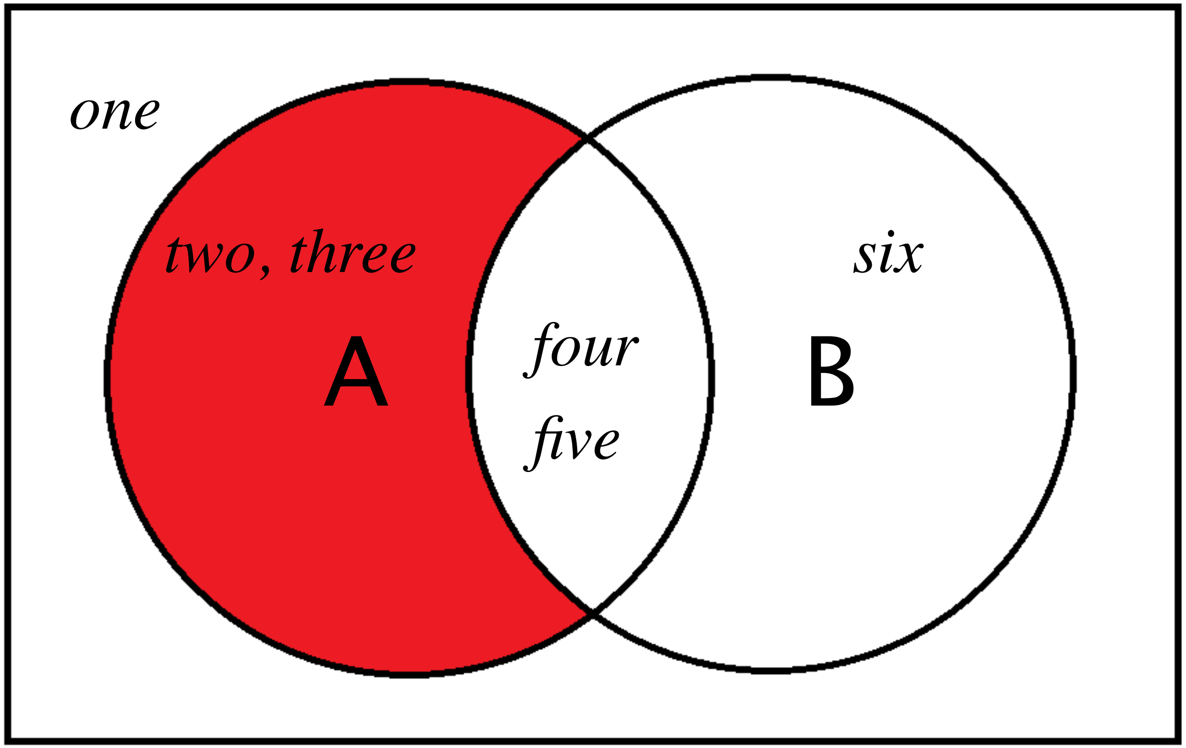
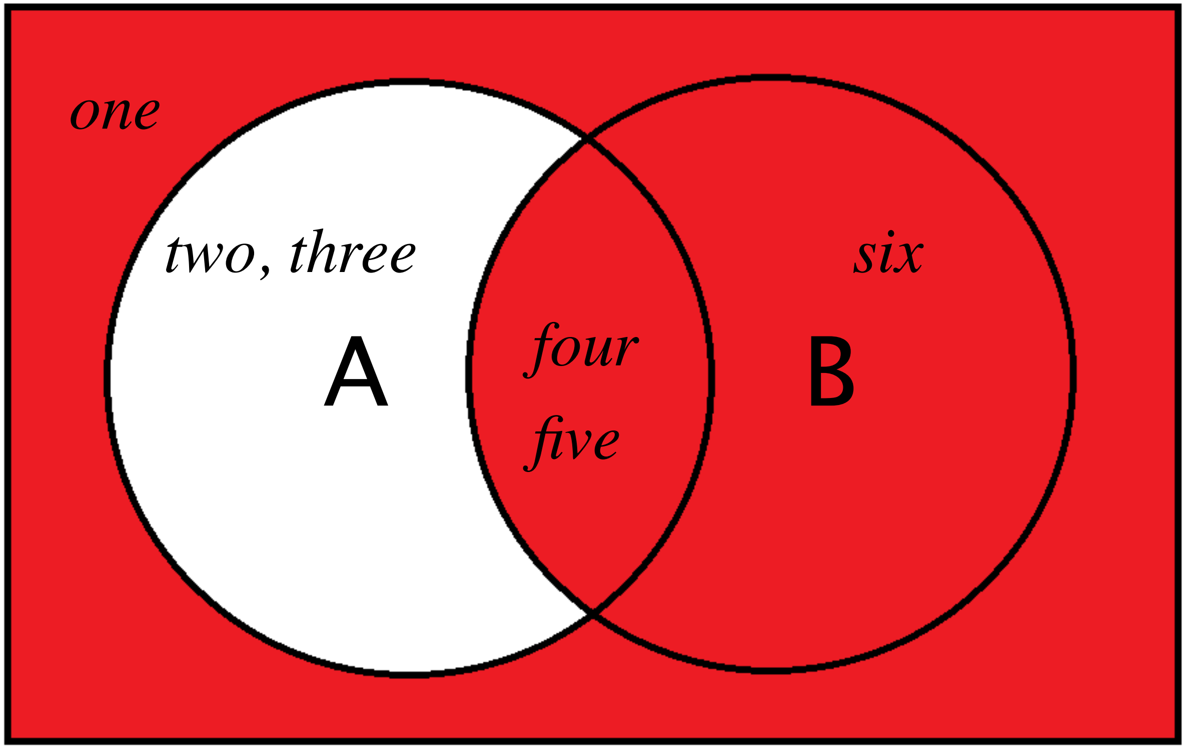


$$A \setminus B = \{two, three\}$$

Set Notation

Venn Diagram:



$$(A \setminus B)' = \{one, four, five, six\}$$

Probability Theory

To consider Probabilities, we need:

1. Sample space: Ω

2. Event space: $\mathcal{F}$

3. Probability measure: P

Probability Theory

To consider Probabilities, we need:

1. Sample space: Ω – the set of all possible outcomes

$$\Omega = \{heads, tails\}$$

$$\Omega = \{one, two, three, four, five, six\}$$

Probability Theory

To consider Probabilities, we need:

2. Event space: $\mathcal{F}$ – the set of all possible events

$$\Omega = \{heads, tails\}$$

$$\mathcal{F} = \{\{heads, tails\}, \{heads\}, \{tails\}, \varnothing\}$$

Probability Theory

To consider Probabilities, we need:

3. Probability measure: *P* $P : \mathcal{F} \rightarrow [0,1]$

P must satisfy two axioms:

$$P(\Omega) = 1$$ Probability of any outcome is 1 (100% chance)

$$P(\bigcup_i A_i) = \sum_i P(A_i)$$ If and only if $A_1, A_2, \ldots$ are disjoint

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$$P(\{one, two\}) = P(\{one\}) + P(\{two\})$$

$$\frac{1}{3} = \frac{1}{6} + \frac{1}{6}$$

Probability Theory

To consider Probabilities, we need:

1. Sample space: Ω
2. Event space: $\mathcal{F}$
3. Probability measure: P

As such, a *Probability Space* is the triple: $(\Omega, \mathcal{F}, P)$

Probability Theory

To consider Probabilities, we need:

The triple: $(\Omega, \mathcal{F}, P)$

i.e. we need to know:

1. The set of potential outcomes;
2. The set of potential events that may occur; and
3. The probabilities associated with occurrence of those events.

Probability Theory

Notable properties of a Probability Space $(\Omega, \mathcal{F}, P)$:

Probability Theory

Notable properties of a Probability Space $(\Omega, \mathcal{F}, P)$:

$$P(A') = 1 - P(A)$$

$$A = \{one, two\}$$

$$A' = \{three, four, five, six\}$$

$$P(A) = 1/3$$

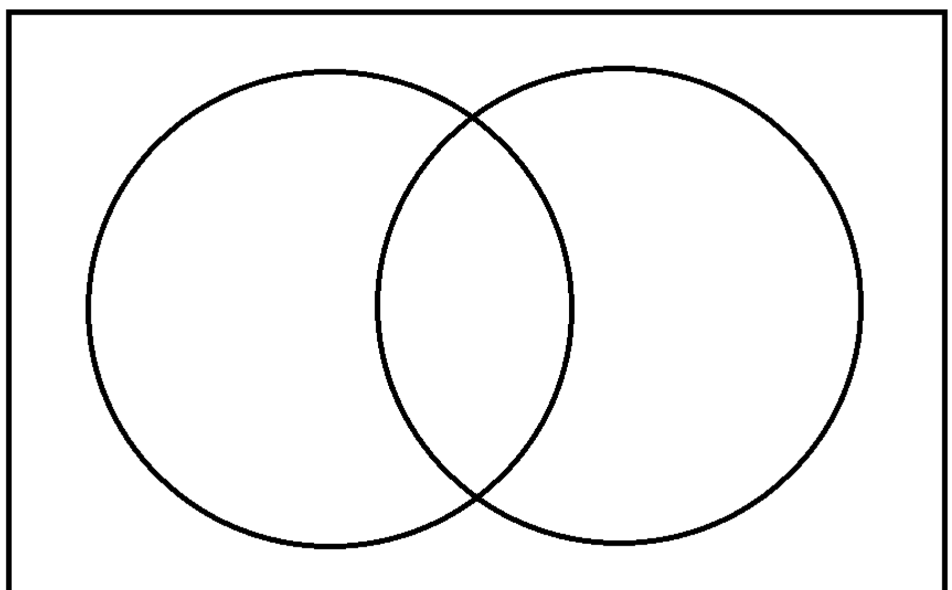
$$P(A') = 2/3$$

Probability Theory

Notable properties of a Probability Space $(\Omega, \mathcal{F}, P)$:

$$P(A') = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



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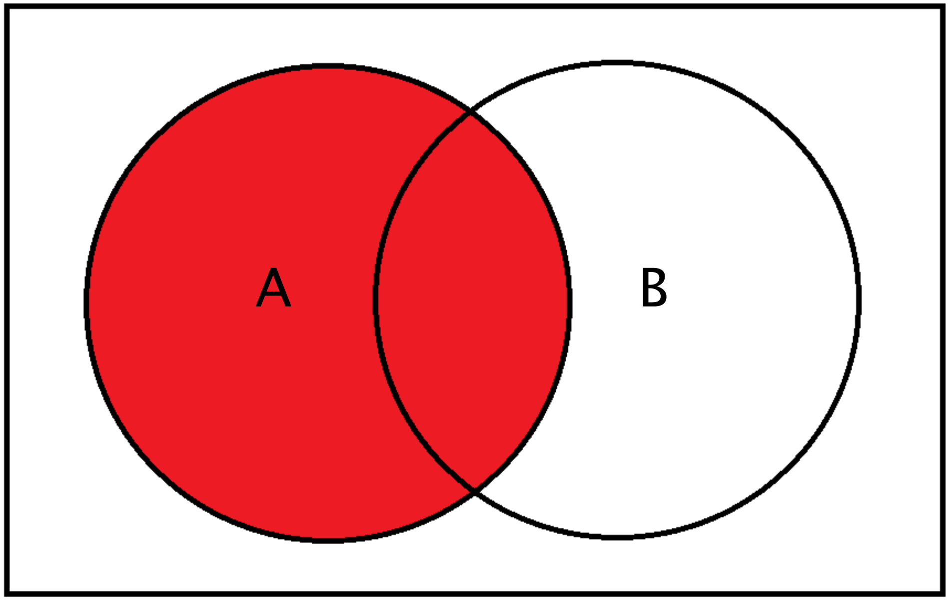

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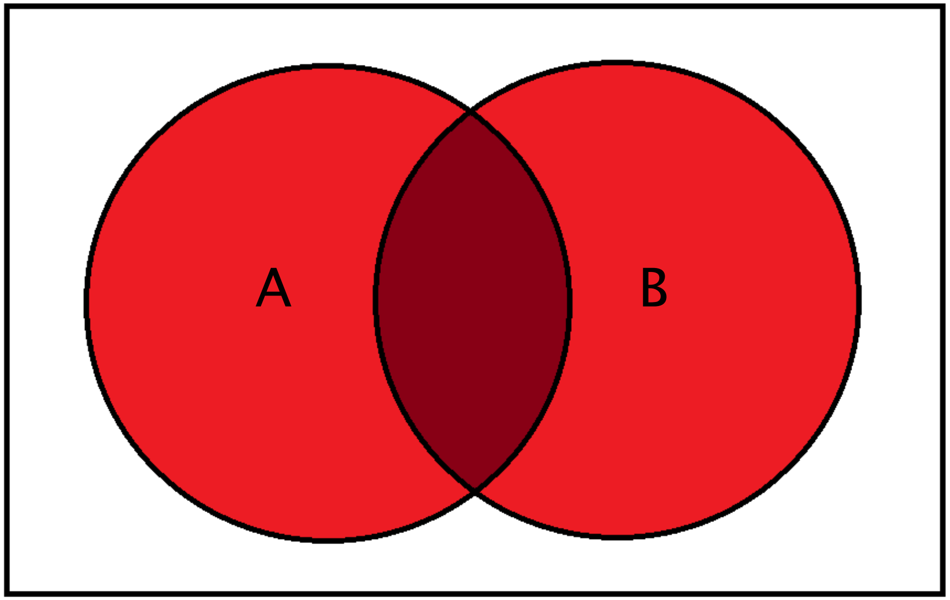

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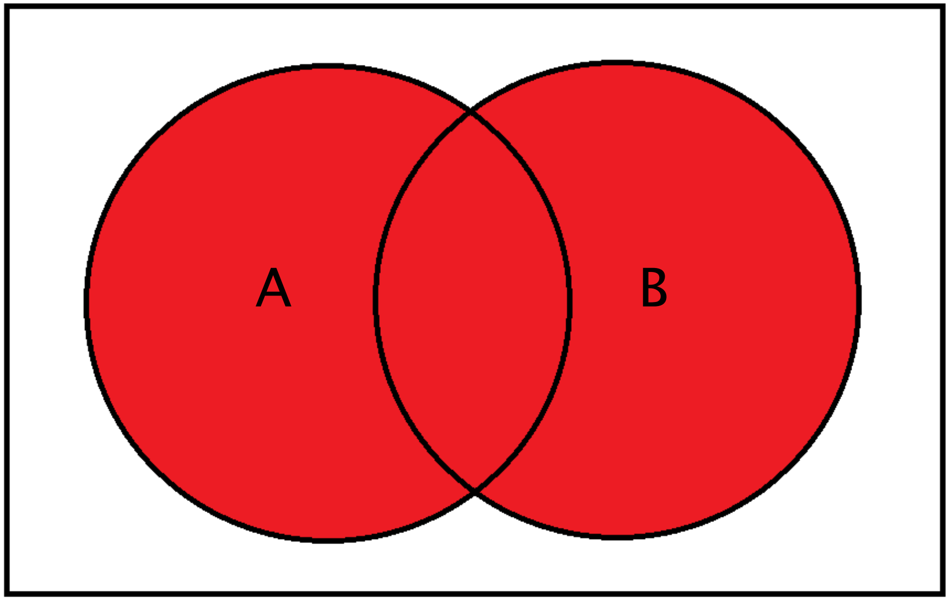

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$$A = \{one, two\} \qquad P(A) = 1/3$$

$$B = \{two, three\} \qquad P(B) = 1/3$$

$$A \cup B = \{one, two, three\} \qquad P(A \cup B) = 1/2$$

$$A \cap B = \{two\} \qquad P(A \cap B) = 1/6$$

Probability Theory

Notable properties of a Probability Space $(\Omega, \mathcal{F}, P)$:

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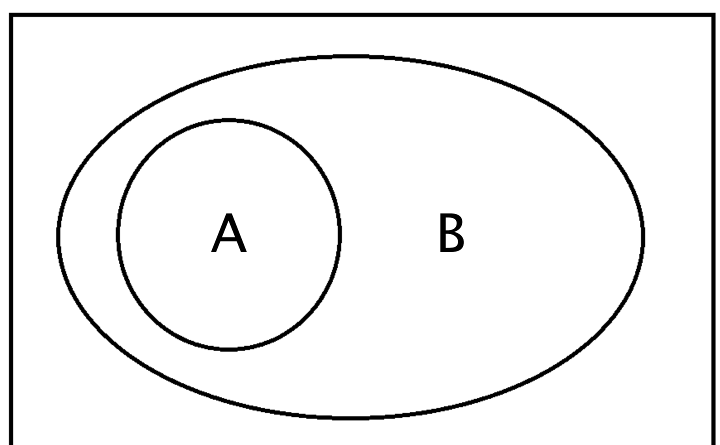
If $A \subseteq B$ then $P(A) \leq P(B)$ and $P(B \setminus A) = P(B) - P(A)$



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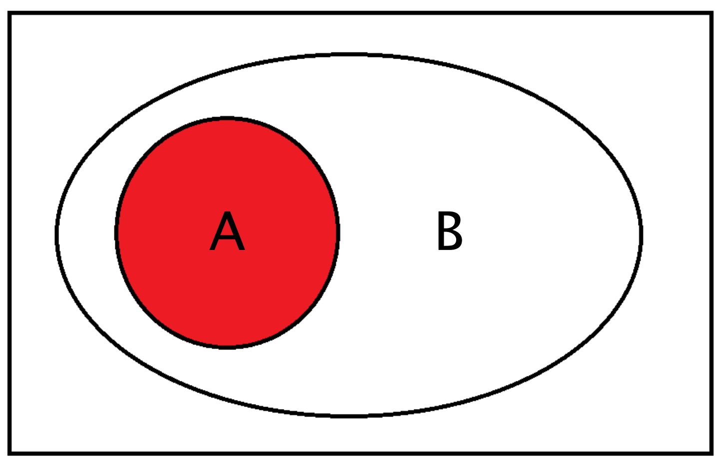

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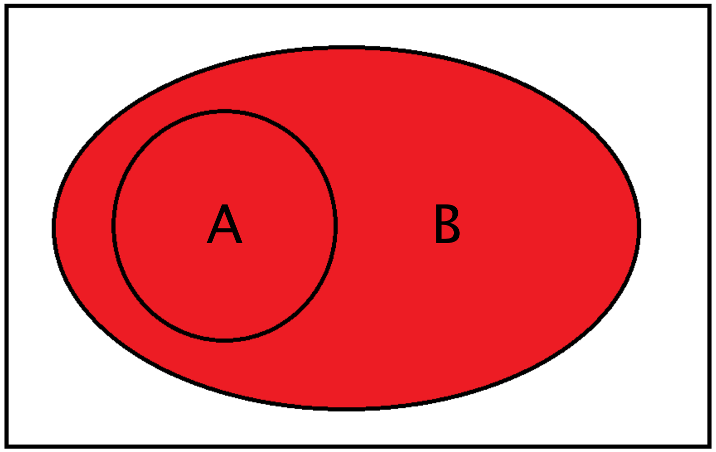

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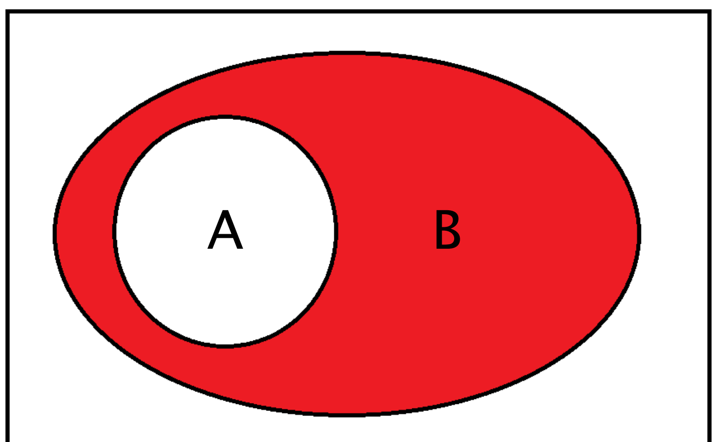

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If $A \subseteq B$ then $P(A) \leq P(B)$ and $P(B \setminus A) = P(B) - P(A)$

$A = \{one, two\}$ $\quad P(A) = 1/3$

$B = \{one, two, three\}$ $\quad P(B) = 1/2$

$B \setminus A = \{three\}$ $\quad P(B \setminus A) = 1/6$

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$$P(\emptyset) = 0$$

Probability Theory

So where's this all going? These examples are trivial!

Probability Theory

So where's this all going? These examples are trivial!

Suppose there are three bags, B_1, B_2 and B_3, each of which contain a number of coloured balls:

- B_1 – 2 red and 4 white
- B_2 – 1 red and 2 white
- B_3 – 5 red and 4 white

A ball is randomly removed from one the bags.

The bags were selected with probability:

- $P(B_1) = 1/3$
- $P(B_2) = 5/12$
- $P(B_3) = 1/4$

What is the probability that the ball came from B_1, given it is red?

Probability Theory

Conditional probability: $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Partition Theorem: $$P(A) = \sum_i P(A \cap B_i)$$ If the B_i partition A

Bayes' Theorem: $$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Random Variables

A *Random Variable* is an object whose value is determined by chance, i.e. random events

Maps elements of Ω onto real numbers, with corresponding probabilities as specified by P

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Probability that the random variable X adopts a particular value x:

$$P(\{w \in \Omega : X(w) = x\})$$

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Shorthand: $P(X = x)$

Random Variables

Example:

If the result is *heads* then WIN – X takes the value 1
If the result is *tails* then LOSE – X takes the value 0

$$\Omega = \{heads, tails\}$$

$$X : \Omega \rightarrow \{0,1\}$$

$$P(X = x) = \begin{cases} P(\{heads\}) & x = 1 \\ P(\{tails\}) & x = 0 \end{cases}$$

$$P(X = x) = 1/2 \qquad x \in \{0,1\}$$

Random Variables

Example:

$$\Omega = \{one, two, three, four, five, six\}$$

Win £20 on a six, nothing on four/five, lose £10 on one/two/three

$$X : \Omega \rightarrow \{-10, 0, 20\}$$

Random Variables

Example:

$$\Omega = \{one, two, three, four, five, six\}$$

Win £20 on a six, nothing on four/five, lose £10 on one/two/three

$$X : \Omega \rightarrow \{-10, 0, 20\}$$

$$P(X = x) = \begin{cases} P(\{six\}) = 1/6 & x = 20 \\ P(\{four, five\}) = 1/3 & x = 0 \\ P(\{one, two, three\}) = 1/2 & x = -10 \end{cases}$$

Note – we are considering the probabilities of events in $\mathcal{F}$

Probability Mass Functions

Given a random variable:

$$X:\Omega \rightarrow A$$

The *Probability Mass Function* is defined as:

$$p_X(x) = P(X = x)$$

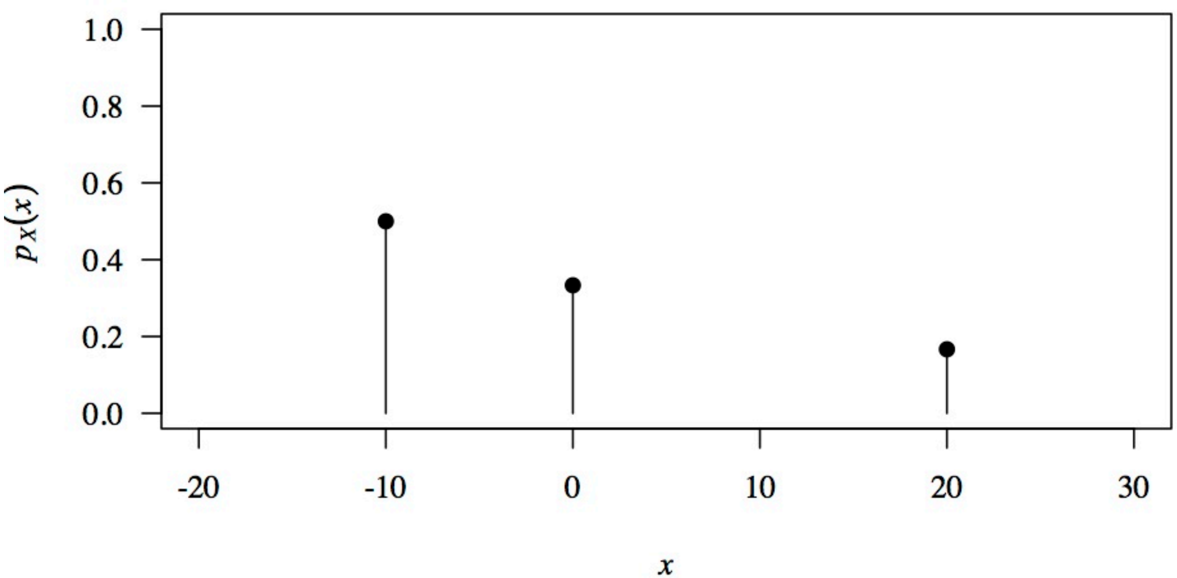
Only for discrete random variables

Probability Mass Functions

Example:

Win £20 on a six, nothing on four/five, lose £10 on one/two/three

$$p_X(x) = \begin{cases} P(\{six\}) = 1/6 & x = 20 \\ P(\{four, five\}) = 1/3 & x = 0 \\ P(\{one, two, three\}) = 1/2 & x = -10 \end{cases}$$

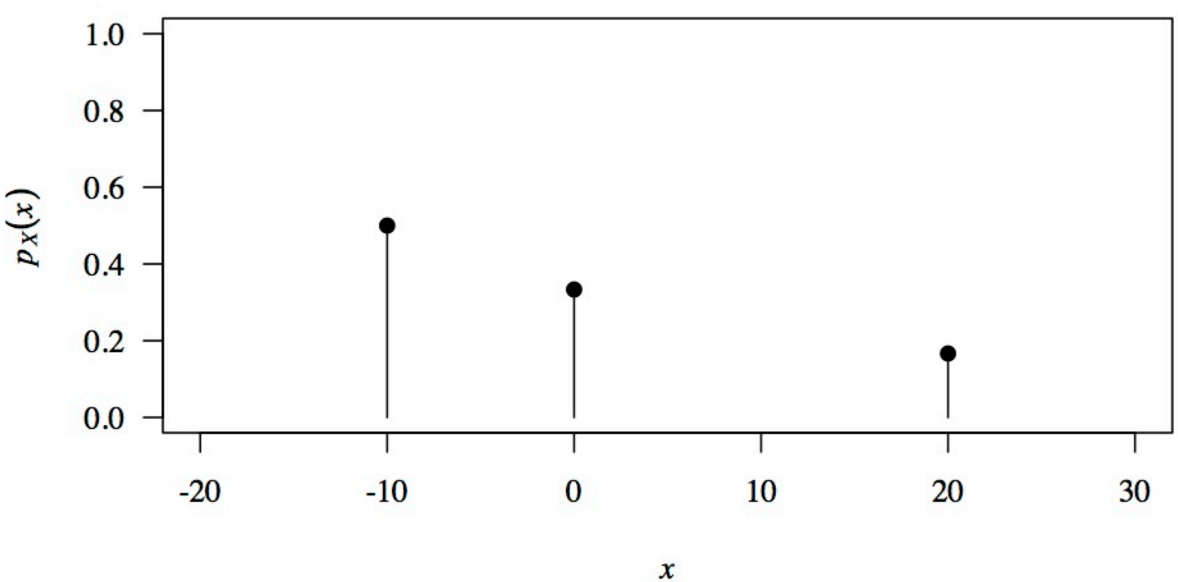


Probability Mass Functions

Notable properties of Probability Mass Functions:

$$p_X(x) \geq 0$$

$$\sum_{x \in A} p_X(x) = 1$$



Probability Mass Functions

Notable properties of Probability Mass Functions:

$$p_X(x) \geq 0$$

$$\sum_{x \in A} p_X(x) = 1$$

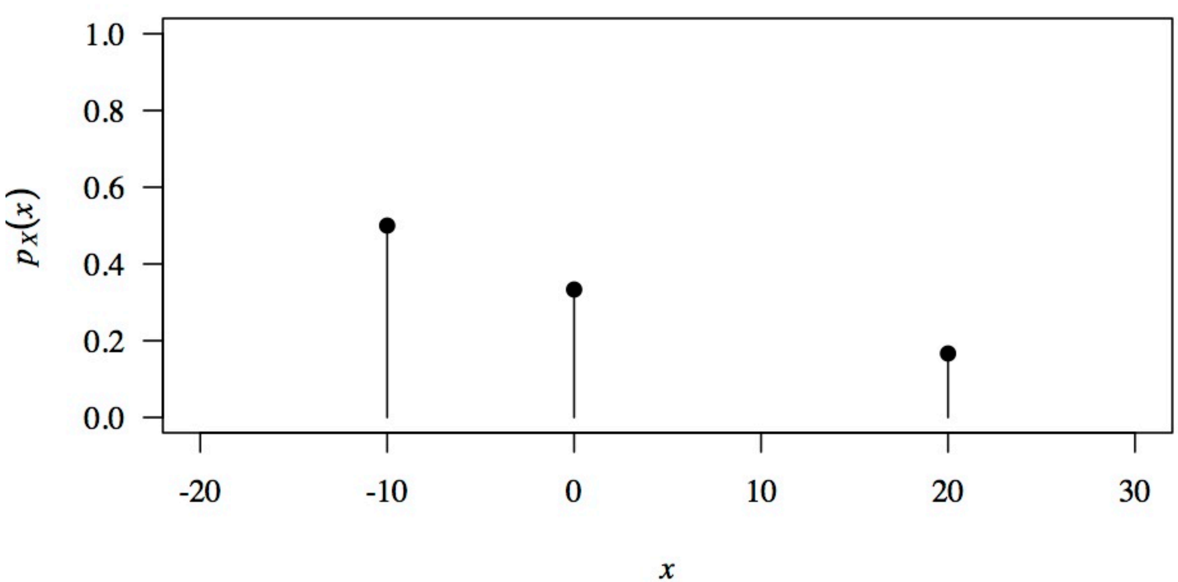
Interesting note:

If *p*() is some function that has the above two properties, then it is the mass function of some random variable…

Probability Mass Functions

For a random variable $X : \Omega \rightarrow A$

Mean: $$E(X) = \sum_{x \in A} x p_X(x)$$



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For a random variable $X : \Omega \to A$

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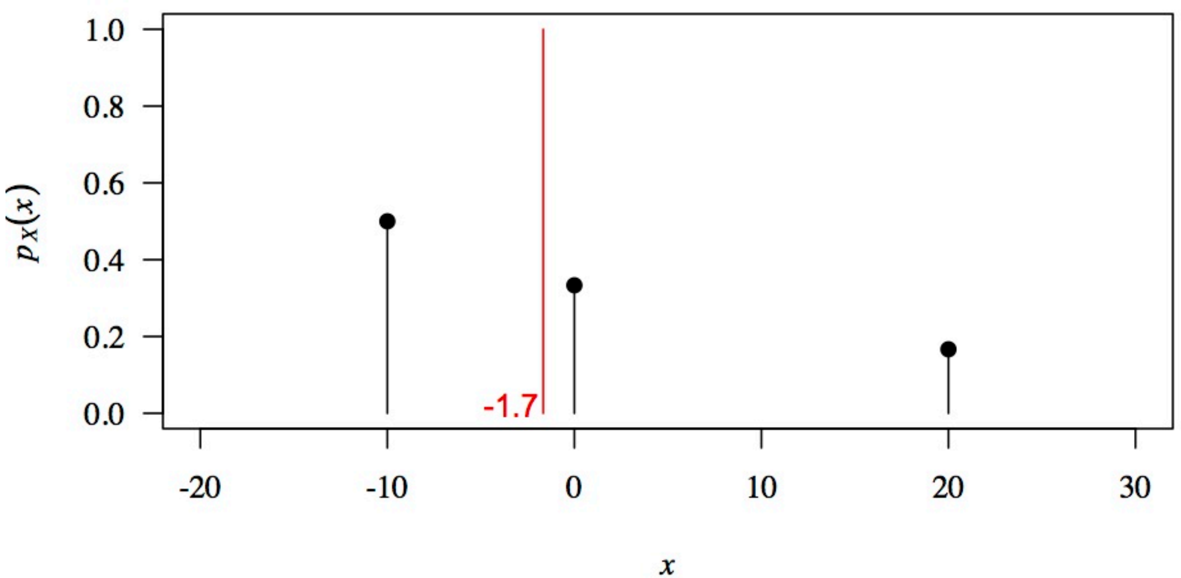


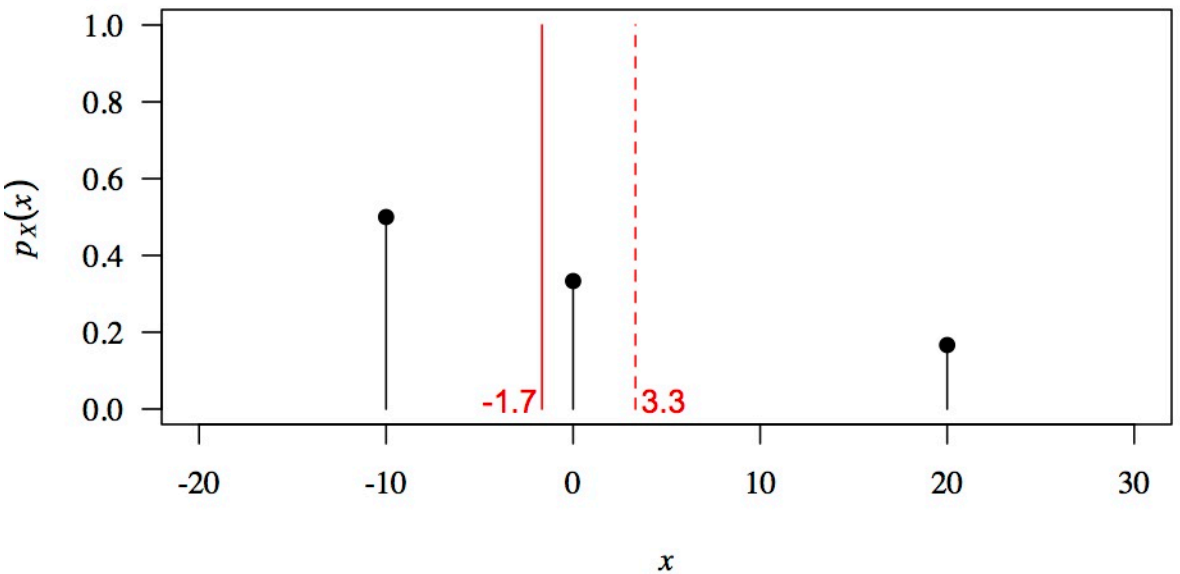
Mean

Probability Mass Functions

For a random variable $X : \Omega \to A$

Mean: $$E(X) = \sum_{x \in A} x p_X(x)$$

Compare with: $$\frac{1}{n}\sum_{i=1}^{n} x_i$$



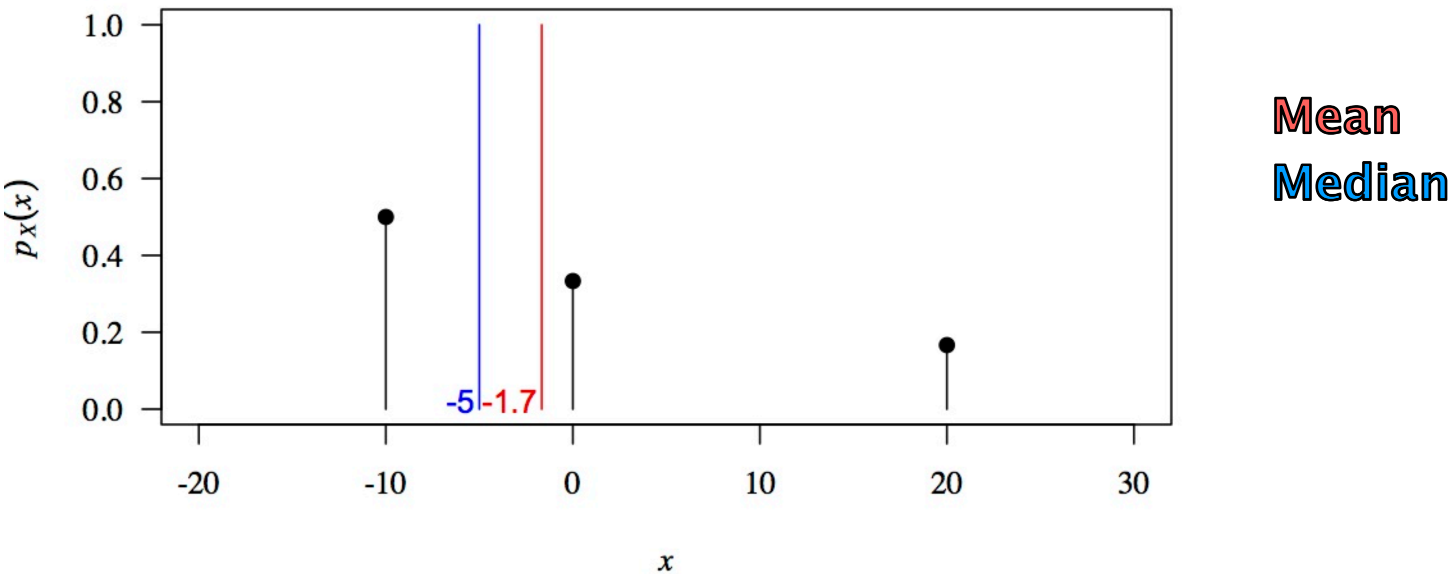
Mean

Probability Mass Functions

For a random variable $X : \Omega \rightarrow A$

Mean: $E(X) = \sum_{x \in A} x p_X(x)$

Median: any m such that: $\sum_{x \leq m} p_X(x) \geq 1/2$ and $\sum_{x \geq m} p_X(x) \geq 1/2$



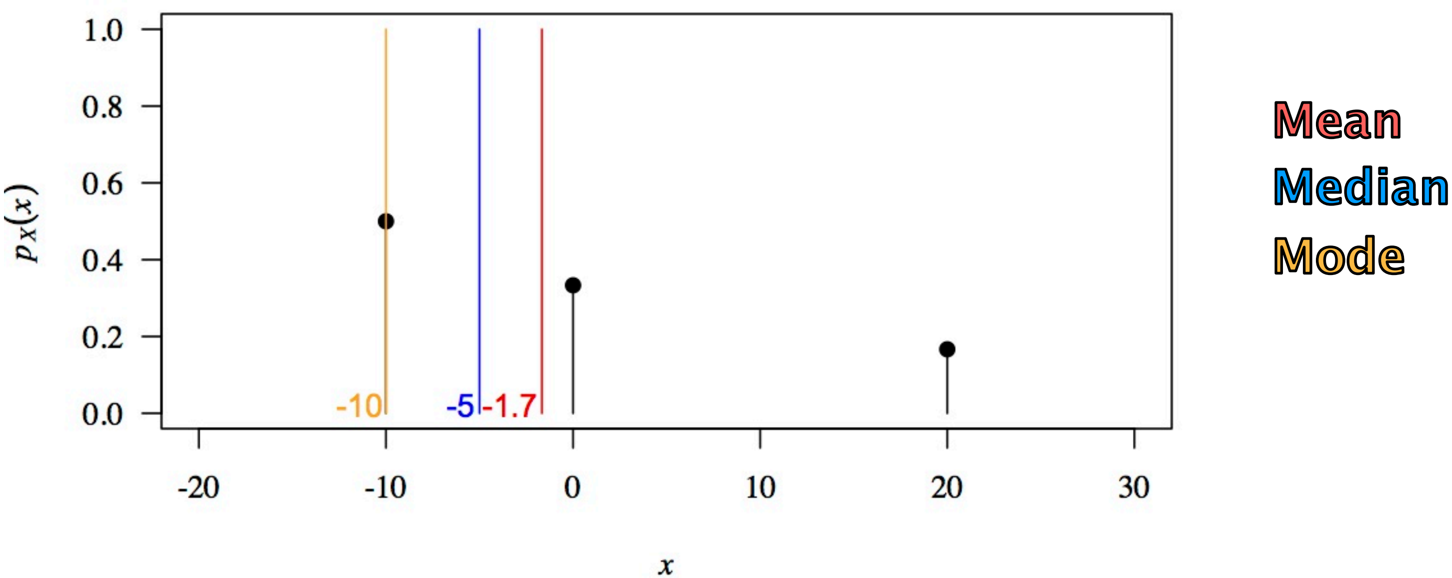
Probability Mass Functions

For a random variable $X : \Omega \rightarrow A$

Mean: $$E(X) = \sum_{x \in A} x p_X(x)$$

Median: any m such that: $\sum_{x \leq m} p_X(x) \geq 1/2$ and $\sum_{x \geq m} p_X(x) \geq 1/2$

Mode: $\underset{x}{\operatorname{argmax}} \left(p_X(x) \right)$: most likely value



Common Discrete Distributions

Common Discrete Distributions

The **Bernoulli** Distribution: $X \sim \mathrm{Bern}(p)$

p : success probability

$$X : \Omega \rightarrow \{0,1\}$$

$$p_X(x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

Common Discrete Distributions

The **Bernoulli** Distribution: $X \sim \mathrm{Bern}(p)$

p : success probability

$$X : \Omega \rightarrow \{0,1\}$$

$$p_X(x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$$

Example:

$$X : \{heads, tails\} \rightarrow \{0,1\}$$

$$p_X(x) = 1/2 \qquad x \in \{0,1\}$$

Therefore $X \sim \mathrm{Bern}(1/2)$

Common Discrete Distributions

The **Binomial** Distribution: $X \sim \mathrm{Bin}(n, p)$ $E(X) = np$

n : number of independent trials

p : success probability

$$X : \Omega \rightarrow \{0, 1, \ldots, n\}$$

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Common Discrete Distributions

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$p_X(x)$: probability of getting x successes out of n trials

p^x : probability of x successes

$(1-p)^{n-x}$: probability of *(n-x)* failures

$\binom{n}{x} = \frac{n!}{x!(n-x)!}$: number of ways to achieve x successes and *(n-x)* failures (Binomial coefficient)

Common Discrete Distributions

The **Binomial** Distribution: $X \sim \text{Bin}(n, p)$ $\quad E(X) = np$

n : number of independent trials

p : success probability

$$X : \Omega \rightarrow \{0, 1, \ldots, n\} \qquad p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$n = 1 : \qquad p_X(x) = p^x (1-p)^{1-x} = \begin{cases} p & x = 1 \\ 1-p & x = 0 \end{cases}$$

$$X \sim \text{Bin}(1, p) \quad \Leftrightarrow \quad X \sim \text{Bern(p)}$$

Common Discrete Distributions

The **Binomial** Distribution: $X \sim \mathrm{Bin}(n, p)$ $E(X) = np$

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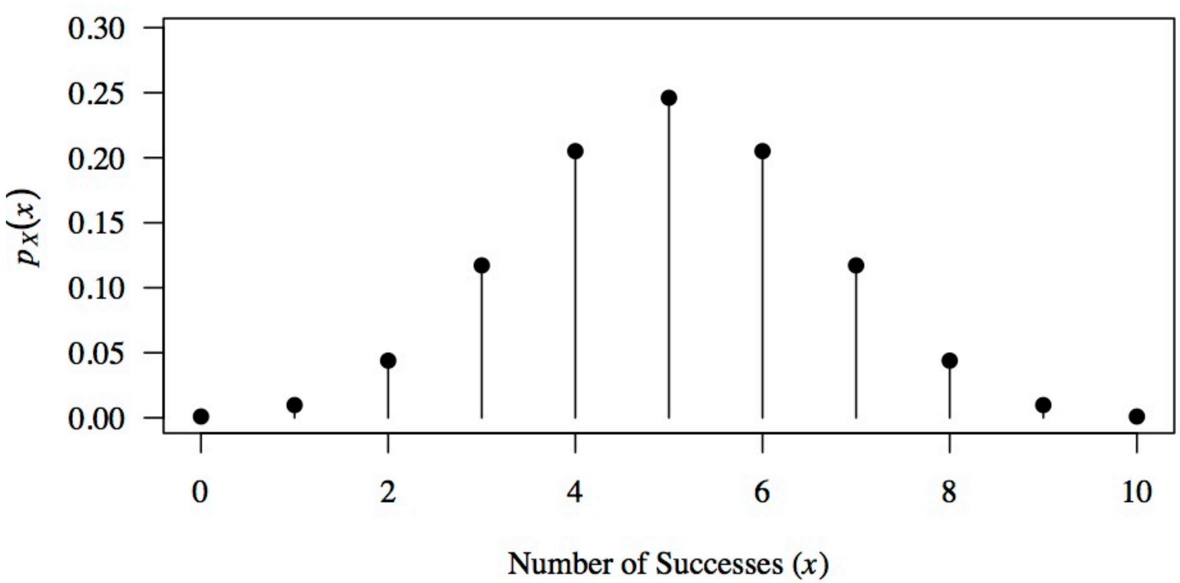

$n = 10$

$p = 0.5$

Common Discrete Distributions

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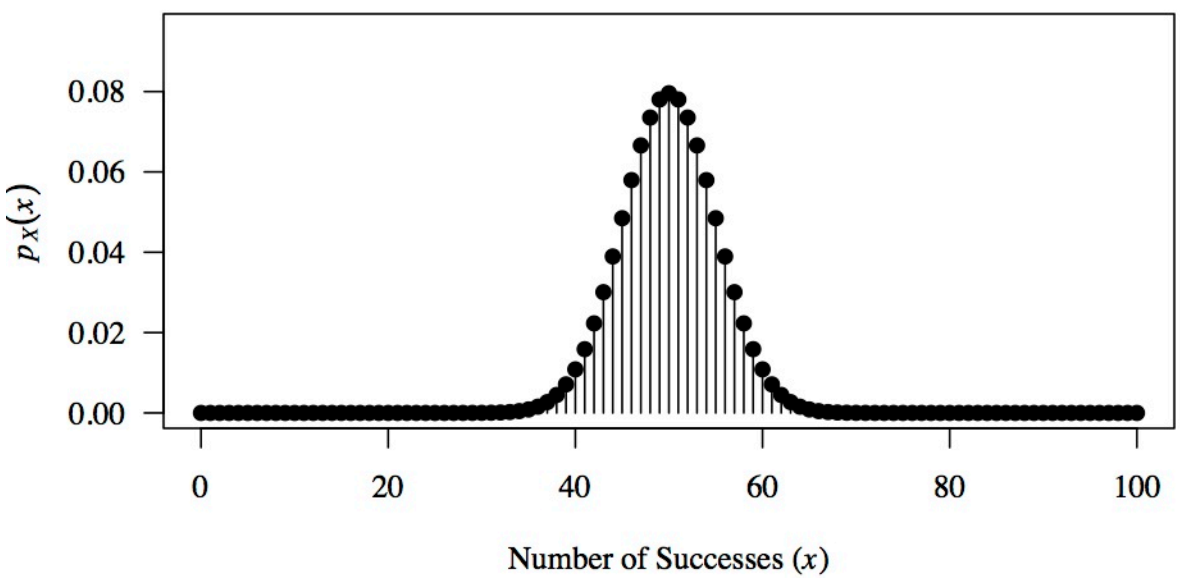

$n = 100$

$p = 0.5$

Common Discrete Distributions

The **Binomial** Distribution: $X \sim \mathrm{Bin}(n, p)$ $E(X) = np$

n : number of independent trials

p : success probability

$$X : \Omega \rightarrow \{0,1,...,n\}$$

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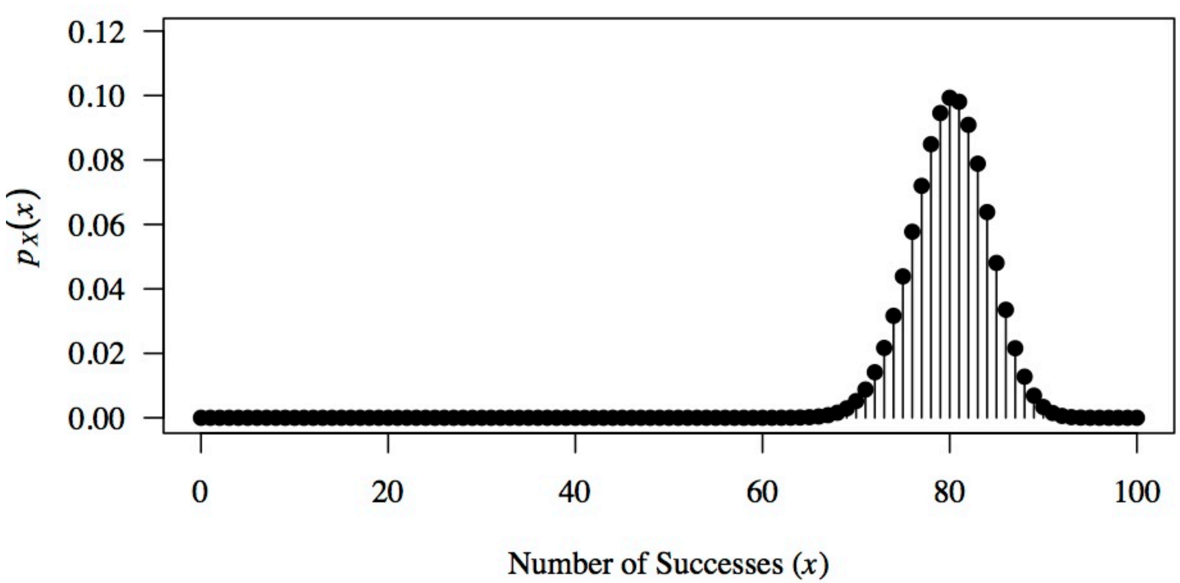

$n = 100$

$p = 0.8$

Common Discrete Distributions

Example:

Number of heads in n fair coin toss trials

$$X : \Omega \rightarrow \{0,1,...,n\}$$

$n = 2$ $\quad\quad$ $\Omega = \{heads : heads, heads : tails, tails : heads, tails : tails\}$

In general: $\quad |\Omega| = 2^n$

Common Discrete Distributions

Example:

Number of heads in n fair coin toss trials

$$X : \Omega \rightarrow \{0,1,...,n\}$$

$n = 2$ $\quad\Omega = \{heads : heads, heads : tails, tails : heads, tails : tails\}$

In general: $\quad |\Omega| = 2^n$

Notice: $\quad X \sim \mathrm{Bin}(n, 1/2)$

$$p_X(x) = \binom{n}{x} 0.5^n \qquad E(X) = n/2$$

Common Discrete Distributions

The **Poisson** Distribution: $X \sim Pois(\lambda)$ $E(X) = \lambda$

Used to model the number of occurrences of an event
that occur within a particular interval of time and/or space

λ : average number of counts (controls rarity of events)

$$X : \Omega \rightarrow \{0,1,...\}$$

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Common Discrete Distributions

The **Poisson** Distribution:

- Want to know the distribution of the number of occurrences of an event $\Rightarrow$ Binomial?

- However, don't know how many trials are performed – could be infinite!

- But we do know the average rate of occurrence: $E(X) = \lambda$

$$X \sim \mathrm{Bin}(n, p) \quad \Rightarrow \quad E(X) = np$$

$$\Rightarrow \quad \lambda = np$$

$$\Rightarrow \quad p = \frac{\lambda}{n}$$

Common Discrete Distributions

Binomial: $$p_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$p = \frac{\lambda}{n} \quad \Rightarrow \quad p_X(x) = \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

Common Discrete Distributions

Binomial: $$p_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$p = \frac{\lambda}{n} \quad \Rightarrow \quad p_X(x) = \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$p_X(x) = \frac{\lambda^x}{x!} \frac{n!}{n^x (n-x)!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

Common Discrete Distributions

Binomial: $$p_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$p = \frac{\lambda}{n} \quad \Rightarrow \quad p_X(x) = \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$p_X(x) = \frac{\lambda^x}{x!} \underbrace{\frac{n!}{n^x (n-x)!}}_{\to 1} \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\underbrace{\left(1 - \frac{\lambda}{n}\right)^x}_{\to 1}} \qquad \text{as } n \to \infty$$

Common Discrete Distributions

Binomial: $$p_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

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$$p_X(x) = \frac{\lambda^x}{x!} \underbrace{\frac{n!}{n^x (n-x)!}}_{\to 1} \frac{\left(1 - \frac{\lambda}{n} \right)^n}{\underbrace{\left(1 - \frac{\lambda}{n} \right)^x}_{\to 1}} \qquad \text{as } n \to \infty$$

$$p_X(x) = \lim_{n \to \infty} \left(\frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n} \right)^n \right)$$

Common Discrete Distributions

Binomial: $$p_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

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$$p_X(x) = \frac{\lambda^x}{x!} \underbrace{\frac{n!}{n^x (n-x)!}}_{\to 1} \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\underbrace{\left(1 - \frac{\lambda}{n}\right)^x}_{\to 1}} \qquad \text{as } n \to \infty$$

$$p_X(x) = \lim_{n \to \infty} \left(\frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^n \right) \qquad = \frac{\lambda^x}{x!} e^{-\lambda}$$

$\square$

Common Discrete Distributions

The Poisson distribution is the Binomial distribution as $n \to \infty$

If $X_n \sim \text{Bin}(n, p)$ then $X_n \xrightarrow{d} \text{Pois}(np)$

If n is large and p is small then the Binomial distribution can be approximated using the Poisson distribution

This is referred to as the:

- "Poisson Limit Theorem"
- "Poisson Approximation to the Binomial"
- "Law of Rare Events"

λ: fixed $\quad n \to \infty \quad \Rightarrow \quad p \to 0$

Poisson is often more computationally convenient than Binomial

References

Countless books + online resources!

Probability theory and distributions:

- Grimmett and Stirzker (2001) Probability and Random Processes. Oxford University Press.

General comprehensive introduction to (almost) everything mathematics (including a bit of probability theory):

- Garrity (2002) All the mathematics you missed: but need to know for graduate school. Cambridge University Press.