Tutorial 3: Probability Theory, Random Variables and Distributions – Answers Sheet

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MRC LMB Statistics Course 2014

Task 1

1.

$$P(A) = 1/4$$

2.

$$P(B) = 6/13$$

3.

$$P(A \cap B) = 3/26$$

4.

$$P(A \cap B) \neq 0$$

so A and B are not mutually exclusive.

5.

$$P(A)P(B) = 1/4 \times 6/13 = 3/26 = P(A \cap B)$$

so A and B are independent.

6.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/4 + 6/13 - 3/26 = 31/52$$

7.

$$\begin{aligned} P(A \cap B') &= P(A \cap (\Omega \backslash B)) \\ &= P((A \cap \Omega) \backslash (A \cap B)) \\ &= P(A \backslash (A \cap B)) \\ &= P(A) - P(A \cap B)) \\ &= 1/4 - 3/26 \\ &= 7/52 \end{aligned}$$

Task 2

1.

$$P(A \cap B) == 0$$

so:

$$\begin{aligned} x = P(B) &= P(A \cup B) - P(A) + P(A \cap B) \\ &= 0.8 - 0.4 + 0 \\ &= 0.4 \end{aligned}$$

2.

From Task 1.7:

$$\begin{aligned} P(A \backslash B) &= P(A \cap B') \\ &= P(A) - P(A \cap B) \end{aligned}$$

so:

$$\begin{aligned} x = P(B) &= P(A \cup B) - P(A) + P(A \cap B) \\ &= P(A \cup B) - P(A \backslash B) \\ &= 0.8 - 0.2 \\ &= 0.6 \end{aligned}$$

3.

$$\begin{aligned} P((A \cap B')') &= 1 - P(A \cap B') \\ &= 1 - P(A \backslash B) \end{aligned}$$

so:

$$\begin{aligned} P(A \backslash B) &= 1 - P((A \cap B')') \\ &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

And thus the problem is reduced to that of Task 2.2, so $x = 0.6$.

4.

From Task 1.7:

$$\begin{aligned} P(A' \cup B) &= P(A') + P(B) - P(A' \cap B) \\ &= 1 - P(A) + P(B) - (P(B) - P(A \cap B)) \\ &= 1 - P(A) + P(A \cap B) \end{aligned}$$

so:

$$\begin{aligned} x = P(B) &= P(A \cup B) - P(A) + P(A \cap B) \\ &= P(A \cup B) - P(A) + (P(A' \cup B) - 1 + P(A)) \\ &= P(A \cup B) + P(A' \cup B) - 1 \\ &= 0.8 + 0.8 - 1 \\ &= 0.6 \end{aligned}$$

5.

A and B are independent, so:

$$\begin{aligned} P(A)P(B) &= P(A \cap B) \\ &= P(A) + P(B) - P(A \cup B) \end{aligned}$$

Which rearranges to:

$$P(B)\,(P(A) - 1) = P(A) - P(A \cup B)$$

$$\begin{aligned} P(B) &= \frac{P(A) - P(A \cup B)}{P(A) - 1} \\ &= \frac{0.4 - 0.8}{0.4 - 1} \\ &= \frac{-0.4}{-0.6} \\ &= 2/3 \end{aligned}$$

6.

$$\begin{aligned} P((A \cap B') \cup (B \cap A')) &= P(A \cap B') + P(B \cap A') \\ &= (P(A) - P(A \cap B)) + (P(B) - P(A \cap B)) \\ &= P(A) + P(B) - 2P(A \cap B) \\ &= P(A) + P(B) - 2P(A)P(B) \\ &= 0.4 + \frac{2}{3} - \left(2 \times 0.4 \times \frac{2}{3}\right) \\ &= 8/15 \end{aligned}$$

Task 3

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A) &= 3P(B) \end{aligned}$$

So:

$$\begin{aligned} 0.7 = P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \\ &= 3P(B) + P(B) - 3P(B)^2 \\ &= 4P(B) - 3P(B)^2 \\ &= 4x - 3x^2 \end{aligned}$$

Rearranging this yields a quadratic equation, which can be solved for x:

$$\begin{aligned} 3x^2 - 4x + 0.7 = 0 \\ x &= \frac{-(-4) \pm \sqrt{(-4)^2 - (4 \times 3 \times 0.7)}}{2 \times 3} \\ &= \frac{4 \pm \sqrt{7.6}}{6} \\ &= 0.207 \qquad \text{(3sf)} \end{aligned}$$

Note that there is only one solution to the problem, as x is only defined on $[0, 1]$ – the solution $x = 1.13$ (3sf) does not exist, as probabilities cannot be greater than 1.

Task 4

1.

From the partition theorem:

$$P(red) = \sum_i P(red \cap B_i)$$

From the definition of conditional probability:

$$P(red \cap B_i) = P(red|B_i)P(B_i)$$

Combining these:

$$\begin{aligned} P(red) &= \sum_i P(red|B_i)P(B_i) \\ &= P(red|B_1)P(B_1) + P(red|B_2)P(B_2) \\ &= 2/5 \times 1/3 + 2/3 \times 2/3 \\ &= 2/15 + 4/9 \\ &= 0.578 \qquad \text{(3sf)} \end{aligned}$$

2.

From Bayes' Theorem:

$$\begin{aligned} P(B_1|red) &= \frac{P(red|B_1)P(B_1)}{P(red)} \\ &= \frac{2/5 \times 1/3}{0.578} \\ &= 0.231 \qquad \text{(3sf)} \end{aligned}$$

3.

$$\begin{aligned} P(B_2|red) &= 1 - P(B_1|red) \\ &= 1 - 0.231 \\ &= 0.769 \qquad \text{(3sf)} \end{aligned}$$

Task 5

1.

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

2.

$$A = \{HHT, HTH, THH\}$$

3.

$$P(A) = 3/8$$

4.

$$P(A) = 3/8$$

5.

```
choose(20,10)
```

Answer: 184756

6.

```
dbinom(10,20,0.5)
```

Answer: 0.176 (3sf)

Task 6

1.

$$X \sim \text{Bin}(10, 0.5)$$

$$\begin{aligned} P(X \geq 8) &= 1 - P(X < 8) \\ &= 1 - P(X \leq 7) \end{aligned}$$

```
1-pbinom(7,10,0.8)
```

Answer: 0.678 (3sf)

2.

$$X \sim \text{Bin}(20, 0.5)$$

$$\begin{aligned} P(X \geq 16) &= 1 - P(X < 16) \\ &= 1 - P(X \leq 15) \end{aligned}$$

```
1-pbinom(15,20,0.8)
```

Answer: 0.630 (3sf)

Task 7

$$X \sim \text{Bin}(95, 0.514)$$

$$\begin{aligned} P(X \geq 60) &= 1 - P(X < 60) \\ &= 1 - P(X \leq 59) \end{aligned}$$

```
1-pbinom(59,95,0.514)
```

Answer: 0.0139 (3sf)

Given the available information, we determine that that there is only a 1.39% chance of no less than 60 boys being born in a given month. Consequently, we conclude that this observation is extraordinary.

Task 8

1.

```
x=0:10
y=dbinom(x,10,0.5)
plot(y~x,pch=19,las=1)
for(i in 1:length(x)){lines(c(1,1)*x[i],c(0,y[i]))}
```

2.

```
plot_binom = function(n,p){
x=0:n
y=dbinom(x,n,p)
plot(y~x,pch=19,las=1)
for(i in 1:length(x)){lines(c(1,1)*x[i],c(0,y[i]))}
}
```